

Chapter 3 Interpolation

§ Lagrange's interpolation

To find a polynomial to approximate the function $f(x)$:

1. Given two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ can generate a linear function as

$$y = ax + b$$

2. Given three points $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$ can generate a quadratic function as

$$y = ax^2 + bx + c$$

3. Given $n+1$ points $(x_0, f(x_0)) \dots (x_n, f(x_n))$ can generate a polynomial of order n

$$y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0.$$

$$y(x) = \sum_{i=0}^n L_{n,i}(x) f(x_i), \quad L_{n,i}(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

§ Divided differences

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0) \dots (x - x_{n-1})$$

Notations

$$f[x_i] = f(x_i), \quad f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$a_0 = f[x_0], \quad a_1 = f[x_0, x_1], \quad a_2 = f[x_0, x_1, x_2], \dots, \quad a_n = f[x_0, \dots, x_n]$$

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n](x - x_0) \dots (x - x_{n-1})$$

Suppose $h = x_{i+1} - x_i$, $x = x_i + sh$

$$p_n(x) = f[x_0] + shf[x_0, x_1] + \dots + s(s-1)(s-n+1)h^n f[x_0, x_2, \dots, x_n]$$

Further, $f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \triangleq \frac{1}{h} \Delta f(x_0),$

$$f[x_0, x_1, x_2] = \frac{1}{x_2 - x_0} \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] = \frac{1}{2h} \left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h} \right]$$

$$\triangleq \frac{1}{2h^2} \Delta^2 f(x_0),$$

$$f[x_0, \dots, x_k] = \frac{1}{k! h^k} \Delta^k f(x_0)$$

So, Newton forward-difference formula

$$p_n(x) = f[x_1] + \sum_{k=1}^n \binom{s}{k} \Delta^k f(x_0), \quad \binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{k!}$$

§ Hermite polynomial

$$\{x_i, f(x_i), f'(x_i)\}_{i=0}^{i=n}$$

$$H_{2n+1}(x) = \sum_{j=0}^n f(x_j) H_{n,j}(x) + \sum_{j=0}^n f'(x_j) \hat{H}_{n,j}(x)$$

Where

$$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L_{n,j}^2(x), \quad \hat{H}_{n,j}(x) = (x - x_j)L_{n,j}^2(x)$$

The divided-difference form of the Hermite polynomial is

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, \dots, z_k](x - z_0)\dots(x - z_{k-1})$$

Where $z_{2k} = z_{2k+1} = x_k$ and $f[z_{2k}, z_{2k+1}] = f'(x_k)$

Ex. $\{x_i, f(x_i), f'(x_i)\}_{i=0}^1$

Notations

$$z_0 = z_1 = x_0, \quad z_2 = z_3 = x_1, \quad l = x_1 - x_0$$

$$H_3(x) = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) \\ + f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2)$$

$$f[z_0] = f(x_0), \quad f[z_0, z_1] = f'(x_0),$$

$$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{l}, \quad f[z_2, z_3] = f'(x_1),$$

$$\begin{aligned} f[z_0, z_1, z_2] &= \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0} = \frac{1}{x_1 - x_0} \left\{ \frac{f(x_1) - f(x_0)}{l} - f'(x_0) \right\}, \\ &= \frac{1}{l} \left\{ \frac{f(x_1) - f(x_0)}{l} - f'(x_0) \right\} \end{aligned}$$

$$\begin{aligned} f[z_1, z_2, z_3] &= \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1} = \frac{1}{x_1 - x_0} \left\{ f'(x_1) - \frac{f(x_1) - f(x_0)}{l} \right\} \\ &= \frac{1}{l} \left\{ f'(x_1) - \frac{f(x_1) - f(x_0)}{l} \right\} \end{aligned}$$

$$\begin{aligned} f[z_0, z_1, z_2, z_3] &= \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{z_3 - z_0} = \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{l} \\ &= \frac{1}{l^2} \left\{ f'(x_1) - 2 \frac{f(x_1) - f(x_0)}{l} + f'(x_0) \right\} \end{aligned}$$

$$H_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{l} \left\{ \frac{f(x_1) - f(x_0)}{l} - f'(x_0) \right\} (x - x_0)(x - x_0)$$

$$+ \frac{1}{l^2} \left\{ f'(x_0) - 2 \frac{f(x_1) - f(x_0)}{l} + f'(x_1) \right\} (x - x_0)(x - x_0)(x - x_1)$$

$$H_3(x) = f(x_0) \left[1 - \frac{(x - x_0)^2}{l^2} + 2 \frac{(x - x_0)^2 (x - x_1)}{l^3} \right]$$

$$+ f'(x_0) \left[(x - x_0) - \frac{(x - x_0)^2}{l} + \frac{(x - x_0)^2 (x - x_1)}{l^2} \right] + f(x_1) \left\{ \frac{(x - x_0)^2}{l^2} \right.$$

$$\left. + f(x_1) \left[\frac{(x - x_0)^2}{l^2} - 2 \frac{(x - x_0)^2 (x - x_1)}{l^3} \right] + f'(x_1) \left\{ \frac{(x - x_0)^2 (x - x_1)}{l^2} \right\} \right\}$$

$$H_3(x) = f(x_0) \left[1 - 3 \frac{(x - x_0)^2}{l^2} + 2 \frac{(x - x_0)^3}{l^3} \right]$$

$$+ f'(x_0) \left[(x - x_0) - 2 \frac{(x - x_0)^2}{l} + \frac{(x - x_0)^3}{l^3} \right]$$

$$+ f(x_1) \left[3 \frac{(x - x_0)^2}{l^2} - 2 \frac{(x - x_0)^3}{l^3} \right] + f'(x_1) \left\{ - \frac{(x - x_0)^2}{l} + \frac{(x - x_0)^3}{l^2} \right\}$$

HW3

1. Use the Lagrange interpolating polynomials of degree one, two, three

and four to approximate $\cos(0.750) = 0.7317$ if $\cos(0.698) = 0.7661$,

$\cos(0.733) = 0.7432$, $\cos(0.768) = 0.7193$, $\cos(0.803) = 0.6946$.

Find the error bound.

2. Use iterated inverse interpolation to find an approximation to the

solution $x - e^{-x} = 0$ using the data $e^{-0.3} = 0.740818$, $e^{-0.4} = 0.670320$,

$e^{-0.5} = 0.606531$, $e^{-0.6} = 0.548812$.

3. A car travelling along a straight road is clocked at a number of points.

The data from the observations are given in the following table, where the time T is in seconds, the distance D is in feet, and the speed V is in feet per second.

T	0	3	5	8	13
D	0	200	375	620	990
V	75	77	80	74	72

- Use a Hermite polynomial to predict the position of the car and its speed when $t = 10$ s.
- Use the derivative of the Hermite polynomial to determine whether the car ever exceeds a 55 mi/h speed limit on the road. If so, what is the first time the car exceeds this speed?
- What is the predicted maximum speed for the car ?